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Endogenous party platforms: ‘stochastic’ membership

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Abstract We propose a model of endogenous party platforms with stochastic membership. The parties’ proposals depend on their membership, while the membership depends both on the proposals of the parties and on the unobserved idiosyncratic preferences of citizens over parties. An equilibrium of the model obtains when the members of each party prefer the proposal of the party to which they belong to, rather than the proposal of the other party. We prove the existence of such an equilibrium and study its qualitative properties. For the cases in which parties use either the average or the median to aggregate the preferences of their members, we show that if the unobserved idiosyncratic characteristics of the parties are similar, then parties make different proposals in the stable equilibria. Conversely, we argue that if parties differ substantially in their unobserved idiosyncratic characteristics, then the unique equilibrium is convergent.

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1 Introduction

The issue of party platform formation has been a subject of substantial attention in political economy. The major idea in this literature is that platforms of political parties are formed in response to preferences of their members, whereas the memberships themselves are, at least in part, determined by the platforms. Thus, in equilibrium the party platforms should respond to the preferences of the members attracted by them. An early paper putting forward an equilibrium concept in which party ideology and its membership are endogenously determined was [Baron \(1993\)](#). His equilibrium concept was related to the one used in the ‘voting with one’s feet’ models developed in the study of local public goods (see [Caplin and Nalebuff \(1997\)](#) for an abstract framework that covers both the political economy and public finance applications). In related work, [Aldrich \(1983a, b\)](#), [Gerber and Ortuño-Ortín \(1998\)](#), and [Poutvaara \(2003\)](#) have considered the interrelationship between partisan policy platforms and political activism.

A major objective in this literature has been establishing conditions for existence of *divergent* equilibria, in which parties propose different policies and attract members with different policy preferences. In a deterministic model of this type (such as [Ortuño-Ortín and Roemer 1998](#) or [Gomberg et al. 2004](#)) such an equilibrium, if it exists, involves a full sorting of agents in terms of their preferences over the policy space: even minute policy differences between parties induce a unique party choice by almost all citizens (in the party activist literature, along the lines of [Aldrich \(1983a\)](#), where there is a third possibility—that of non-participation—it is still normally assumed that those actually actively taking part in partisan activities do it in the ideologically closest party). However, such perfect sorting is not commonly observed in reality: even ideologically identical people may frequently find themselves in different parties based on idiosyncratic non-policy considerations (perhaps, historical esthetical or personal). These non-policy issues might not even be observable by an outsider, making the observed policy preferences only stochastic predictors of individual party choice. This is, of course, not a new idea in political science, where the study of stochastic models of voting has been widespread for a long time (see [Coughlin \(1992\)](#) for a survey). Our stochastic preference model of endogenous membership follows the same intuition. Our focus is, however, somewhat different. In particular, rather than considering the vote-maximizing parties in an electoral context we restrict our attention to parties aggregating members’ preferences and try to establish to which extent the results of an older deterministic models (such as our own [Gomberg et al. \(2004\)](#)) may be extended to this new setting. In modeling parties as aggregating preferences of their members, while membership is, in turn, determined in part (but not fully) by party policy positions our paper is related to the work by [Roemer \(2001\)](#). Our approach, however, is different in such crucial aspects as, among others, our more explicit modeling of membership decisions, and the nature of intra-party decision

rules (which in Roemer's case discriminate among members of different ideologies based on belonging to a 'partisan core').

The approach incorporating the stochastic party preference has, indeed, been one of those proposed already by [Caplin and Nalebuff \(1997\)](#). They, however, found it unsatisfactory, since, in their opinion, such equilibria could, in many cases, be fully determined by the stochastic preference component and not by the observables. Furthermore, they believed, it could not guarantee existence of equilibria exhibiting policy divergence. For, as they note, 'there is the possibility that as the noise approaches zero, two institutions' positions will approach each other.' They further conjectured that 'whenever there is the nonexistence of an equilibrium without probabilistic choice it must be the case that the group positions approach each other in the probabilistic choice model as the noise goes to zero.' Remarkably, as we show in this paper, the converse is also true: whenever conditions for existence of divergent equilibria in a 'non-stochastic' model, such as those in [Caplin and Nalebuff \(1997\)](#) and [Gomberg et al. \(2004\)](#) hold, such equilibria will also exist if we introduce a stochastic component in individual preferences, as long as the latter is sufficiently small. This, of course, implies that, contrary to what [Caplin and Nalebuff \(1997\)](#) conjectured, noise is not going to fully determine the equilibrium policy positions of parties.

A seemingly major difficulty in this extension is that the previous studies obtained existence of sorting equilibria from the properties of the space of perfectly sorting partitions of the population between parties (see [Caplin and Nalebuff \(1997\)](#) and [Gomberg et al. \(2004\)](#); in the context of local public goods, this approach goes back to [Westhoff \(2005\)](#)). In the absence of perfect sorting, this approach is, of course, not feasible. However, the crucial feature of the deterministic model is, in fact, not the sorting *per se*, but a certain 'instability' of the convergent equilibrium. To see that, suppose there are two parties which propose the same policy. Then, the entire population is indifferent and the vote splits in such a way that a convergent equilibrium obtains. We require now that even minor policy perturbations result in full population sorting and sharply divergent policies. It turns out that, if that is the case, the existence of (at least one) sorting equilibrium is, in fact, guaranteed. In this paper, we show that this intuition, in part, extends to the stochastic context: if the convergent (or 'almost' convergent) equilibrium exists but is unstable to small policy perturbations, it may be used to detect existence of divergent equilibria. In fact, as the addition of the stochastic component adds continuity to the model, in a sense the results become more transparent in this setting. In particular, in a context of a "generalized example" in our framework, we show that when parties are perceived by voters to be very similar in non-policy terms, so that the observed randomness of individual partisan choice is relatively small, the results of the deterministic model extend to the stochastic case.

For the moment (and for simplicity) we abstract from possible strategic electoral competition by parties (in the terminology of [Caplin and Nalebuff \(1997\)](#) our parties are "membership-based"). The main reason here is methodological: we believe that the issue of endogenizing party membership is distinct from the issue of strategic behavior by party leaders in a democratic election. Our main concern here is the former, and we want to consider it separately. This assumption may be viewed as appropriate for either a model of parties in a setting without commitment (*e.g.*, when voters would not believe a party, once in office, can implement policies not supported by its membership) or

in a setting without true electoral competition (*e.g.*, if parties' share of the office is determined through non-electoral means). Of course, we do intend to explore extending our results to cover the case of possible strategic interactions between parties.

We assume that a political party is characterized by a policy position and by some exogenously fixed idiosyncratic non-policy position or characteristic. The policy position will be endogenously determined by the membership of the party. Thus, parties are represented by positions in a multi-dimensional space with a fixed position in one dimension (the non-policy dimension) and a variable position in the other dimensions. [Krasa and Polborn \(2010\)](#) and [Krasa and Polborn \(2012\)](#) study equilibrium in a multi-dimensional model in which each candidate is exogenously fixed on some dimension. Unlike in our case, their candidates are office-motivated and their equilibrium concept is quite different from the one we consider here (in our case parties are ideological and the ideology is endogenously determined). [Dziubiński and Roy \(2011\)](#) consider a model of electoral competition in a two-dimensional policy space where the position in one of the dimensions is fixed. They analyze existence of convergent and divergent Nash equilibria. A difference between the models is that their parties are Downsian (non-ideological, but concerned with election), whereas our main goal is to analyze the endogenous formation of party ideologies.¹

Our model generates interesting predictions on the relation between the policy proposals of parties and their idiosyncratic non-policy characteristics. It is often claimed that when ideological parties strongly differ in non-policy characteristics (that are exogenously given) they have more incentives to propose divergent policies (see [Roe-mer 2011](#)). This is due to the fact that proposing a more 'radical' policy is not that costly for a party, since voters' decisions are very much influenced by the large differences in the non-policy variable. In our model, however, this does not need to be the case. If agents' preferences over the policy variables are independent of their preferences over the non-policy characteristic of the parties, increasing the differences between those non-policy characteristics might yield convergence of the policy proposals². The intuition is clear: If parties are very different in their non-policy characteristics, their membership is basically determined by such non-policy characteristic. Hence,

¹ As already mentioned, our theoretical model makes the following two assumptions: that the policy proposed by each political party can be seen as an aggregation of the policy preferences of its members, and that party membership is determined by both policy and non-policy characteristics of parties. In an appendix to the unpublished working paper version of the paper (see [Gomberg et al. 2013](#)), we provide some empirical evidence, suggesting that these might be realistic assumptions. In particular, we first analyze the political platforms of the main parties and the average ideal policy of their supporters. We find that, in countries with only two major parties, such as US and UK, the political platform of the party and the average ideal policy of its supporters are strikingly similar. This is not the case in other countries with more than two major parties or in a clearly unstable political period. In the same appendix, we also analyze data about the self-reported ideological position of citizens and which party they vote for. As expected, the ideological position is not sufficient to perfectly determine the voting behavior of citizens (some left-wing citizens vote for the right-wing party, and some right-wing citizens vote for the left-wing party). Moreover, we show that such data seem consistent with the existence of a non-policy variable such that agents' preferences over it are independent of their preferences on the left-right ideological space.

² The authors in [Dziubiński and Roy \(2011\)](#) provide a somehow related result. In their case, if parties strongly differ in their fixed policies in a given dimension, there is full convergence in the other dimension. In our case, in general there is no full convergence and the explanation and logic behind our result is very different from theirs.

unless preferences in the policy and non-policy dimensions are correlated (in which case sorting in the non-policy dimension would by itself impose policy divergence), members of the two parties will be quite similar regarding their preferences on the policy variables. And, since parties just aggregate the preferences of their members, their policy proposals will be very similar.

The rest of this paper is organized as follows: Section 2 presents the model and develops a general existence result. Sections 3 and 4 present the results for, respectively, the mean and median voter rules in a single dimension of policy space, and Sect. 5 concludes.

2 Model

There are two parties. Party $j = 1, 2$ proposes a policy vector $x_j \in X$, where X is a non-empty compact and convex subset of \mathbb{R}^n with non-empty interior. In addition to a policy x_j , party $j = 1, 2$ is characterized by a non-policy variable $y_j \in Y$. The set Y is assumed to be a closed interval of \mathbb{R} . It may be interpreted as reflecting currently fixed or intrinsic characteristics that matter for individual preferences. We shall make the following assumption.³

Assumption 1 $y_1 < y_2$.

There is a continuum of agent types with preferences over both policy and non-policy characteristics of parties. Specifically, each agent of type $(\alpha, \beta) \in A \times B \subset \mathbb{R}^n \times \mathbb{R}$ has Euclidean preferences represented by the utility function

$$u(x, y; \alpha, \beta) = -\|(x, y) - (\alpha, \beta)\|$$

where $x \in X$ is the policy platform adopted by the party and $y \in Y$ is the intrinsic characteristic of the party. For simplicity, we shall take $A = X$ and $Y = B$. Thus, for fixed $y \in Y$, an agent of type (α, β) may be identified with his/her ideal policy.

There is a measure space of agents (citizens) $(A \times B, \mathcal{A} \times \mathcal{B}, \eta)$, where \mathcal{A} is the Borel σ -algebra on A , \mathcal{B} is the Borel σ -algebra on B and η is a measure on $\mathcal{A} \times \mathcal{B}$ such that $\eta(A \times B) = 1$. We denote the distribution function of η as $F(\alpha, \beta)$.

Assumption 2 *The measure η is represented by a continuous density function $f(\alpha, \beta)$, which is equivalent to Lebesgue measure.*

Citizens play a twofold role. Each agent is a voter and a member of the party.⁴ Given the parties policy and the intrinsic characteristics (x_1, y_1) and (x_2, y_2) , citizens join the party they like the most. Thus, the individual party choice is unambiguous. However, from the point of view of the parties, the second coordinate of individual type (α, β) is unobservable. Hence, for the parties the observable individual preferences

³ The case $y_1 = y_2$ being the one we considered in Gombert et al. (2004).

⁴ In most real cases, only a small fraction of the population is a member of a party. Our results remain true if we assume that the set of citizens who become members of parties is a random sample of the whole population.

over policies (given by α) may serve only as an imperfect predictor of individual party choice. From the perspective of the party, the citizen's decision which party to join appears stochastic.

A party membership is observed as a finite measure on A . We shall restrict ourselves to measures on A which induce a continuous population density $g(\alpha)$ defined on A . In the following, we will consider only non-null subsets of X . For the sake of concreteness and ignoring zero-measure sets, we will identify the set of possible party memberships with the set Σ consisting of all continuous functions $g : A \rightarrow \mathbb{R}_{++}$ such that $0 < \int_A g(\alpha) d\alpha < 1$. On Σ we consider the topology determined by the supremum norm $\|g\| = \sup\{|g(\alpha)| : \alpha \in A\}$ for $g \in \Sigma$. Our results would not change if one considers the L^1 topology on Σ .

A political party $j = 1, 2$ chooses its policy by aggregating the observed policy preferences of its members, according to some fixed rule $P_j : \Sigma \rightarrow X$, defined for non-null subsets of X . As parties do not observe β , the aggregation applies only to α . We shall denote the profile of party policy-setting rules as $P = (P_1, P_2)$. Therefore, our model is determined by $M = (A, B, f, y_1, y_2, P_1, P_2)$.

Example 3 As an example of such a rule, we may consider the mean (respectively, the median voter rule, defined only for $n = 1$) which assigns to each party the ideal policy of its mean voter (respectively, median voter). These two aggregation rules are studied in Sects. 3 and 4 below. In particular, the mean voter rule P^μ assigns to each admissible population partition $v = (v_1, v_2) \in \Sigma \times \Sigma$ its mean,

$$P_i^\mu(v) = \frac{1}{v_i(A)} \int_A \alpha dv_i(\alpha), \quad i = 1, 2.$$

Consider now the exogenous idiosyncratic party characteristic parameters y_1 and y_2 . For each party $i = 1, 2$ a policy proposal profile $x = (x_1, x_2) \in X \times X$ induces a party membership defined by

$$A_i(x) = \{(\alpha, \beta) \in A \times B : u(x_i, y_i; \alpha, \beta) \geq u(x_j, y_j; \alpha, \beta), j \neq i\}$$

Since we only consider population densities which are absolutely continuous with respect to Lebesgue measure, the line segment $A_1(x) \cap A_2(x)$ has measure zero. Ignoring this zero-measure set, we will think of $A_1(x)$ and $A_2(x)$ as a partition of $A \times B$.

We define next the mapping $\sigma = (\sigma_1, \sigma_2) : X \times X \rightarrow \Sigma \times \Sigma$. Given a proposal $x = (x_1, x_2)$, the induced party memberships $A_1(x)$ and $A_2(x)$ determine measures $\sigma_1(x), \sigma_2(x) \in \Sigma$ whose associated density functions are

$$g_i(\alpha; x) = \int_{\{\beta \in B : (\alpha, \beta) \in A_i(x)\}} f(\alpha, \beta) d\beta, \quad i = 1, 2 \quad (1)$$

That is, for each Lebesgue measurable set $S \subset A$, its measure induced by x is

$$\sigma_i(x)(S) = \int_S g_i(\alpha; x) d\alpha$$

And we define $\sigma(x) = (\sigma_1(x), \sigma_2(x))$. For $n = 1$, we can provide a more explicit formulation of the densities $g_1(\alpha; x)$, $g_2(\alpha; x)$. Let $x = (x_1, x_2) \in X \times X$. Agents affiliate to one or the other party, depending on which side of the line

$$z(t; x) = \frac{x_1 - x_2}{y_2 - y_1}t + \frac{y_1 + y_2}{2} - \frac{x_1^2 - x_2^2}{2(y_2 - y_1)} \quad (2)$$

they lie. That is,

$$g_1(\alpha; x) = \int_{-\infty}^{z(\alpha; x)} f(\alpha, \beta) d\beta, \quad g_2(\alpha; x) = \int_{z(\alpha; x)}^{+\infty} f(\alpha, \beta) d\beta \quad (3)$$

We remark that, even though,

$$\int_A (g_1(\alpha, x) + g_2(\alpha, x)) d\alpha = 1$$

the measures $\sigma_1(x)$ and $\sigma_2(x)$ do not necessarily have disjoint supports. That is, the product $g_1(\alpha, x)g_2(\alpha, x)$ does not necessarily vanish. Thus, the observable characteristic $\alpha \in A$ does not completely describe the behavior of the voters. Even though agents are fully rational, their behavior is stochastic from the point of view of the parties.

Definition 4 A population partition $\nu = (\nu_1, \nu_2)$ shall be considered *admissible* if $\nu_1, \nu_2 \in \Sigma$ and for every $S \in \mathcal{A}$, we have that $\nu_1(S) + \nu_2(S) = \eta(S \times B)$.

The following lemma shows that the mapping $\sigma : X \times X \rightarrow \Sigma \times \Sigma$ is well defined. That is, the functions $g_1(\alpha; x)$ and $g_2(\alpha; x)$ are continuous on the variable $\alpha \in A$. The proof is provided in the ‘‘Appendix’’.

Lemma 5 Suppose Assumptions 1, and 2 hold. Then, the mapping $\sigma_i : X \times X \rightarrow \Sigma$ is continuous for each $i = 1, 2$.

Remark 1 Note that, if $y_1 = y_2$ and, for example, $x_1 < x_2$, the induced population densities are

$$g_1(\alpha; x) = \begin{cases} \int_B f(\alpha, \beta) d\beta & \text{if } \alpha < \frac{x_1 + x_2}{2} \\ \frac{1}{2} \int_B f(\alpha, \beta) d\beta & \text{if } \alpha = \frac{x_1 + x_2}{2} \\ 0 & \text{if } \alpha > \frac{x_1 + x_2}{2} \end{cases}$$

$$g_2(\alpha; x) = \begin{cases} \int_B f(\alpha, \beta) d\beta & \text{if } \alpha > \frac{x_1 + x_2}{2} \\ \frac{1}{2} \int_B f(\alpha, \beta) d\beta & \text{if } \alpha = \frac{x_1 + x_2}{2} \\ 0 & \text{if } \alpha < \frac{x_1 + x_2}{2} \end{cases}$$

Hence, the measures $\sigma_1(x)$ and $\sigma_2(x)$ do have disjoint supports. In particular, there is full sorting of the parties’ members into two disjoint sets and we recover the ‘non-stochastic’ framework posed by [Caplin and Nalebuff \(1997\)](#).

2.1 Equilibrium

Our notion of equilibrium assumes free mobility of the electorate across parties. Thus, our equilibrium concept requires that: (i) the proposals of the parties are determined by their respective membership and (ii) given the party's proposal, all the members of each party prefer their own party to the alternative.

Definition 6 Given the profile of party policy-setting rules P and the idiosyncratic party characteristics y_1, y_2 , we say that $(x^*, v^*) \in X \times X \times \Sigma^2$ is a **multi-party equilibrium** if:

1. $x^* = P(v^*)$
2. $v^* = \sigma(x^*)$

Furthermore, the equilibrium is **divergent** if $x_1^* \neq x_2^*$. Otherwise, we say that the equilibrium is **convergent**.

The above definition captures the idea that, in equilibrium, the platforms of the parties are consistent with the preferences of the members they attract. Consider the mapping $\phi : X \times X \rightarrow X \times X$ defined by $\phi(x) = P(\sigma(x))$. Clearly, an equilibrium is just a fixed point of this mapping.

Assumption 7 (continuity) *The function $P_j : (\Sigma, \|\cdot\|) \rightarrow X$ is continuous for $j = 1, 2$.*

Proposition 8 *Suppose Assumptions 1, 2 and 7 hold. Then, there exists an equilibrium.*

Proof Consider the mapping $\phi : X \times X \rightarrow X \times X$ defined by $\phi(x) = P(\sigma(x))$. The fixed points of this mapping correspond to equilibria of the model. The mapping is clearly defined on the entire space $X \times X$ and continuity follows from Assumption 7 and Lemma 5. As $X \times X$ is compact and convex, by Brouwer's fixed point theorem there must exist at least one (possibly convergent) equilibrium.

Note that if $y_1 = y_2$, so that individual membership is fully determined by policy positions, this model becomes deterministic and fully falls into the framework posed by Caplin and Nalebuff (1997). Therefore, Proposition 8 shows that, as long as what we are interested in is merely existence of multi-party equilibria, the present model still fits the approach of Caplin and Nalebuff (1997) and Gomberg et al. (2004): basic continuity and minimal internal support assumptions on the policy-setting rules (as in Gomberg et al. (2004)), together with the exogenously imposed difference between parties yield existence of equilibrium. The novel case for us in this paper is $y_1 \neq y_2$.

Since we are assuming that $y_1 < y_2$, we trivially obtain full sorting of the agents in the $A \times B$ space. However, this sorting may be entirely caused by the difference in the B dimension. In particular, if the observed policy positions of the parties (that is, their projections onto the A component) coincide, then this equilibrium would still be convergent. Hence, Proposition 8 is silent on the existence of divergent equilibria.

Remark 2 Under Assumption 1, note that if $x_1 = x_2$, then we have that

$$\begin{aligned} g_1(\alpha; x) &= g_1(\alpha) = \int_{-\infty}^{\frac{y_1+y_2}{2}} f(\alpha, \beta) d\beta \\ g_2(\alpha; x) &= g_2(\alpha) = \int_{\frac{y_1+y_2}{2}}^{+\infty} f(\alpha, \beta) d\beta \end{aligned} \quad (4)$$

does not depend on the specific value of $x_1 = x_2$. Hence, the quantities

$$\lambda_i = \sigma_i(x)(A) = \int_A g_i(\alpha; x) d\alpha \quad (5)$$

do no depend either on the particular value of $x_1 = x_2$.

We remark next that there is at most a unique convergent equilibrium. Of course, the following result does not exclude the existence of other divergent equilibria.

Proposition 9 *Suppose Assumptions 1, 2 and 7 hold. Then, the number of convergent equilibria is at most 1.*

Proof Consider a policy profile $x = (x_1, x_1)$. Then, the party membership is described by the densities in (4), which is independent of $x_1 \in [0, 1]$. That is, $P \circ \sigma$ is constant in the diagonal of $A \times A$. It follows that there exists, at most, one convergent equilibrium.

Definition 10 Parties are *ex ante* identical if they use the same aggregating rule, $P = P_1 = P_2$, and $P(\sigma_1(x)) = P(\sigma_2(x))$ whenever $x = (x_1, x_2)$ is such that $x_1 = x_2$.

Trivially, if parties are *ex ante* identical there is a convergent equilibrium. The existence of such equilibrium relies on us being able to support identical policies in equilibrium. Note that even if parties use the same internal policy-setting rule $P_1 = P_2 = P$, this does not imply the existence of a convergent equilibrium. If the induced distributions σ_1 and σ_2 are different, it might happen that $P(\sigma_1(x)) \neq P(\sigma_2(x))$, for a proposal x with $x_1 = x_2$. Consider the following simple example.

Example 11 Let $n = 1$ and suppose the population is distributed over $[0, 1] \times [0, 1]$. Suppose that both parties use the mean voter rule P^μ so that

$$P_i^\mu(v) = \frac{1}{v_i(A)} \int_A \alpha dv_i(\alpha)$$

Consider two possible population distributions over this domain: the uniform distribution over the whole of $[0, 1] \times [0, 1]$, so that individual preferences over the two dimensions are entirely uncorrelated, and the uniform distribution over the diagonal $\{(\alpha, \alpha) : \alpha \in [0, 1]\}$, in which there is perfect correlation between the individual ideal points in the two dimensions. Note that in both cases the unconditional distribution of ideal points in the policy space is uniform: $f_1(\alpha) = 1$. Assume, for simplicity, that $y_2 = 1 - y_1$.

Consider the case $y_1 = y_2 = \frac{1}{2}$. As the citizen preference in this case depends only on $f_1(\alpha)$, for either of the two distributions there are the same three equilibria: $x_1 = x_2 = 1/2$; $x_1 = \frac{1}{4}$ $x_2 = \frac{3}{4}$; and $x_1 = \frac{3}{4}$ $x_2 = \frac{1}{4}$.⁵ As we increase the difference between the idiosyncratic positions y_1 and y_2 , the two cases will become increasingly different. As we show in Section 4, when the individual preferences in the two dimensions are uncorrelated $x_1 = x_2 = \frac{1}{2}$ remains an equilibrium always, while the other two equilibria converge to it, so that the symmetric equilibrium becomes the only one as long as $y_1 \leq \frac{1}{4}$.

In contrast, in the case when preferences in the two dimensions are perfectly correlated, as long as $y_1 > \frac{1}{4}$ it is the divergent equilibria $x_1 = \frac{1}{4}$ $x_2 = \frac{3}{4}$ and $x_1 = \frac{3}{4}$ $x_2 = \frac{1}{4}$ that remain unchanged, while the convergent equilibrium disappears and gradually shifts to $x_1 = y_2$, $x_2 = y_1$. When $y_1 \leq \frac{1}{4}$ so that the parties are far apart, the only surviving equilibrium in this case is $x_1 = \frac{1}{4}$ $x_2 = \frac{3}{4}$. The population partition is now fully determined by the party difference in the y dimension.

In a deterministic version of the model, [Caplin and Nalebuff \(1997\)](#) have postulated the assumption that the party policy rules would never result in identical policies if party populations have opposing preference, in the sense of being divided by a hyperplane in the ideological space. Even in their model this assumption is problematic, unless the policy rules are just aggregating intra-party preferences (“membership-based” in their terminology). And since, in the present model, the sorting is not perfect, it is entirely inapplicable here.

That introducing *ex ante* difference between parties may effectively impose policy divergence exogenously has been a particular concern in earlier work of [Caplin and Nalebuff \(1997\)](#), who found a similar approach to be unsatisfactory for this very reason. However, we will argue that in our model policy divergence obtains even if parties are *ex ante* identical. The main concern in Sects. 3 and 4 below is to establish conditions for existence of divergent equilibria and determine their stability and robustness properties.

3 Mean voter rule

In this section, we focus on the case in which parties aggregate the preferences of their members using the mean voter rule. We will show that, even if parties are *ex ante* identical, there is a divergent equilibrium, at least as long as the difference $|y_2 - y_1|$ is sufficiently small. Our approach allows us to compute explicit bounds on the exogenous interparty difference $|y_2 - y_1|$, which guarantees policy divergence. For simplicity, in computing these bounds, we assume a unidimensional policy space $n = 1$.

⁵ Note that the convergent equilibrium in this example relies on full indifference of every agent between the two parties, requiring a slight modification of the definitions along the lines of [Gomberg et al. \(2004\)](#). In the uncorrelated case, this equilibrium is properly defined for any $y_1 < \frac{1}{2}$. In the correlated case, we rely on complete population indifference in the centrist equilibrium even if $y_1 < \frac{1}{2}$. However, as long as the correlation is imperfect, so that the population distribution is not concentrated on the hyperplane, this would no longer be the case, while, as we show below, the structure of equilibrium set remains the same.

3.1 Existence of divergent equilibria for *ex ante* identical parties

The mean voter rule, P^μ , assigns to each admissible population partition $\nu = (\nu_1, \nu_2) \in \Sigma \times \Sigma$ its mean,

$$P_i^\mu(\nu) = \frac{1}{\nu_i(A)} \int_A \alpha \, d\nu_i(\alpha)$$

For $x \in X \times X$, the induced population partition $\sigma_j(x)$ is represented by the density $g_j(\alpha; x)$ defined in (1). Hence, given the standing proposals $x = (x_1, x_2) \in X \times X$ of the parties, each induced membership of the parties chooses

$$P_j^\mu(\sigma_j(x)) = \frac{\int_A \alpha g_j(\alpha; x) \, d\alpha}{\int_A g_j(\alpha; x) \, d\alpha} = \frac{\int_A \alpha g_j(\alpha; x) \, d\alpha}{\sigma_j(x)(A)}$$

The following Lemma, which is proved in the “Appendix”, states that the mean voter rule satisfies Assumption 7.

Lemma 12 *The map $\phi : X \times X \rightarrow X \times X$, defined by $\phi(x) = P^\mu(\sigma(x))$, is continuous and differentiable.*

Recall that an equilibrium is a fixed point of the mapping $\phi : X \times X \rightarrow X \times X$ defined by $\phi(x) = (P_1^\mu(\sigma_1(x)), P_2^\mu(\sigma_2(x)))$. Therefore, (x_1^*, x_2^*) is an equilibrium if

$$x_j^* = \frac{\int_A \alpha g_j(\alpha; x^*) \, d\alpha}{\int_A g_j(\alpha; x^*) \, d\alpha} \quad j = 1, 2$$

Given a distribution of the population $f: A \times B \rightarrow \mathbb{R}$, let $\mu_j = P_j^\mu(\sigma_j(x))$, $j = 1, 2$ be the means of the parties when parties make the same proposal $x = (x_1, x_2)$ with $x_1 = x_2$. Recall that even if parties make the same proposal, we cannot guarantee that $P_1^\mu(\sigma_1(x)) \neq P_2^\mu(\sigma_2(x))$. So, in general, we do not expect that there will be a convergent equilibrium.

On the other hand, if parties are *ex ante* identical and make the same proposal $x = (x_1, x_2)$, with $x_1 = x_2$, then $P_1^\mu(\sigma_1(x)) = P_2^\mu(\sigma_2(x)) = \mu$, the observed mean of the overall population on A . Therefore, (μ, μ) is a convergent equilibrium. We are interested in finding conditions under which, even when parties are *ex ante* identical, there are other divergent equilibria.

To do so, assume that $\mu_1 = \mu_2 = \mu$. So, (μ, μ) is a convergent equilibrium. We will prove that, provided that y_1 and y_2 are close enough, then the convergent equilibrium (μ, μ) is unstable. The stability of (μ, μ) along the diagonal is immediate, since the function ϕ trivially maps the diagonal into (μ, μ) . To prove the existence of other divergent equilibria, we will determine conditions under which the fixed point (μ, μ) is unstable off-diagonal of $X \times X$. By the Lefschetz’s fixed point theorem (see [Guillemin and Pollack 2010](#), pp. 119–130), the total sum of the indices of the fixed points must be equal to 1 (the Euler characteristic of $X \times X$). Recall that the index of a non-degenerate fixed point may be calculated as $(-1)^d$, where d is the dimension

of the unstable manifold of the fixed point. As the co-dimension of the diagonal is 1, the index of the diagonal fixed point equals $(-1)^n$, which implies it cannot be unique if $n = 1$.

Hence, a divergent equilibrium will exist if we can prove that the convergent equilibrium (μ, μ) is unstable off the diagonal. That is, it will be enough to show that the eigenvalues of the matrix

$$B(x_1, x_2) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} - 1 & \frac{\partial \phi_1}{\partial x_2} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} - 1 \end{pmatrix} \quad (6)$$

have different signs around (μ, μ) . Hence, we only need to show that

$$\lim_{x \rightarrow (\mu, \mu)} \det(B(x)) < 0 \quad (7)$$

As stated in the next result, it turns out that if $|y_2 - y_1|$ is small enough, the above condition can be assured. Therefore, Proposition 13 below shows that if $|y_2 - y_1|$ is small enough, then either there is no convergent equilibrium (because $\mu_1 \neq \mu_2$) or else (when $\mu_1 = \mu_2$) the unique convergent equilibrium is unstable. That is, if $|y_2 - y_1|$ is small enough, the stable equilibria must be divergent. Let λ_1, λ_2 be defined by (5).

Proposition 13 *Let $n = 1$. Suppose that parties use the mean voter rule and that $\mu_1 = \mu_2 = \mu$. If*

$$|y_2 - y_1| < \frac{1}{\lambda_1 \lambda_2} \int_A (\alpha - \mu)^2 f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha \quad (8)$$

then there exists a divergent equilibrium.

The proof is provided in the “Appendix”. In there, we establish that $\det(B(x)) < 0$. The bound established by Proposition 13 depends only on two bits of population statistics: the variance of ideal points in the observable ideological space of those citizens who would be indifferent between parties in the absence of ideological differences between them, and the relative size λ_1 of the part of the population that exogenously prefers one party to another when there is no ideological difference between them. The following is an immediate result of Proposition 13.

Corollary 14 *Let $n = 1$. Suppose that the parties are ex ante identical, use the mean voter rule and the bound in (8) holds. Then, there exists a divergent equilibrium.*

Example 15 Let $n = 1$ and the population distribution be uniform on $[0, 1] \times [0, 1]$ so that $f(\alpha, \beta) = 1$. Suppose that both parties use the mean voter rule P^μ and the idiosyncratic variables are such that $0 < |y_2 - y_1| < 1/(12\lambda_1(1 - \lambda_1)) = 1/3$. By our proposition, there exists a divergent equilibrium. As an example satisfying this inequality take $y_1 = 0.55$ and $y_2 = 0.20$. It is not difficult to show that the unique divergent equilibrium with $x_1 < x_2$ has $x_1 = 0.416$, $x_2 = 0.651$. Of course, there is an additional symmetric equilibrium with $x_1 = 0.651 > x_2 = 0.416$. See Fig. 1.

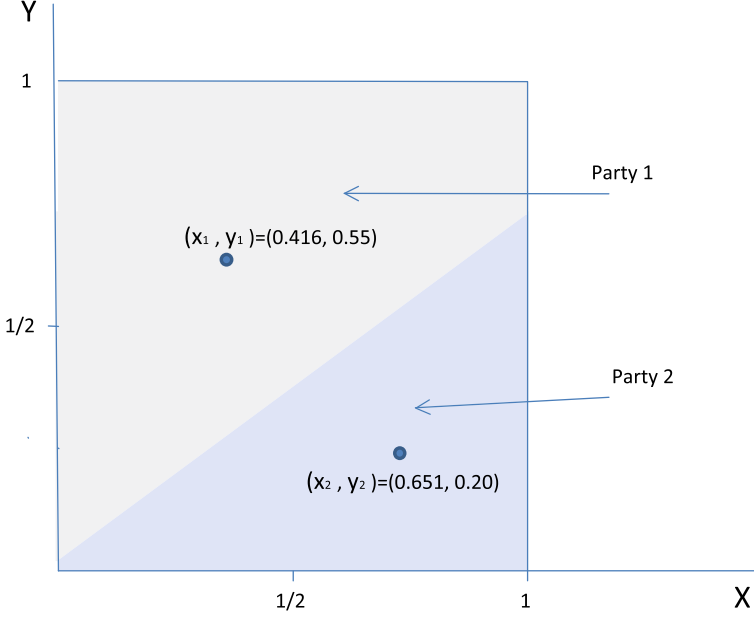


Fig. 1 The divergent equilibrium with $x_1 < x_2$

On the other hand, while, strictly speaking, exceeding the bound does not guarantee the uniqueness of the convergent equilibrium, examples of the latter are not hard to find.

Proposition 13 shows that there is continuity between the deterministic and non-deterministic models: a small amount of uncertainty about individual membership decision does not affect the existence of a divergent equilibrium. Indeed, the Proposition implies that exogenous differences between the parties imply that the citizens' membership decision is mostly determined by the observed policy differences. In particular, if we let $y_2 = y_1$ and $x_1 \neq x_2$, we are reduced to the deterministic model. Hence, the above result shows that the deterministic case is not isolated.

We now address some robustness questions of our model. Proposition 16 below shows that as $|y_1 - y_2|$ gets smaller and we approximate the deterministic case, there will remain a properly divergent equilibrium, in which party positions will stay away of each other. The proof is provided in the “Appendix”.

Let $y_1^s = \frac{1}{2}(y_1 + y_2 + s(y_1 - y_2))$ and $y_2^s = \frac{1}{2}(y_1 + y_2 + s(y_2 - y_1))$. This allows us to define, for each $s \in (0, 1)$ the mapping $z^s(t; x)$ defined by (2), the corresponding induced measures σ^s and the mapping $\phi^s = P \circ \sigma^s$. As before, for each s the fixed points of $\phi^s(x)$ will be equilibria of the corresponding model $M^s = (A, B, f, y_1^s, y_2^s, P_1^\mu, P_2^\mu)$.

Proposition 16 *Assume $n = 1$ and suppose that the parties are ex ante identical and use the mean voter rule. Let y_1, y_2 and f satisfy the bound in (8). Then, for each $s \in (0, 1)$ there is a fixed point \bar{x}^s of $\phi^s(x)$ such that the family $\{\bar{x}^s\}_{s \in (0, 1)}$ does not converge to the diagonal in $X \times X$, as $s \rightarrow 0$.*

3.2 Other ‘mean’-type rules

It should be noted that the continuity result of Proposition 13 can be extended to policy rules other than the mean voter rule, though the precise bound would be, of course, different. In particular, suppose that, instead of choosing the ideal point of the mean of its voter distribution, parties propose policies according to a different rule

$$P_j^h(\eta) = \frac{\int_A h(\alpha) d\eta}{\eta(A)}$$

where $h : A \rightarrow A$ is a non-constant continuous mapping.

To fix ideas, suppose the parties are *ex ante* identical. Let $x_1 = x_2$, so $g(\alpha) = g_1(\alpha; x) = g_2(\alpha; x)$ does not depend on $x = (x_1, x_2)$. Denote the policy society as a whole would adopt as $\chi = \int_A h(\alpha) g(\alpha) d\alpha \in \text{int}(A)$. Following the steps of the proof of Proposition 13, one may easily obtain that the following bound on the exogenous difference between parties guarantees the existence of divergent equilibria:

$$0 < |y_2 - y_1| < \frac{1}{\lambda_1 \lambda_2} \int_A (h(\alpha) - \chi)(\alpha - \chi) f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha$$

where λ_1, λ_2 are defined by (5). This implies that, for these rules, and as long as $\int_A (h(\alpha) - \chi)(\alpha - \chi) f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha > 0$, sufficiently small uncertainty about individual membership choices leads to the existence of divergent equilibria.

3.3 Policy convergence and non-ideological characteristics for the mean voter rule

In this section, we provide robust numerical and theoretical examples showing a type of converse result for Proposition 13, as $|y_2 - y_1|$ gets large. That is, if parties which aggregate preferences via the mean voter rule are very different in their non-policy characteristics, then their policy proposals will turn out to be very similar. Let $f_0 : A \times [-\infty, \infty] \rightarrow \mathbb{R}_+$ be a nonvanishing, continuous, bounded function with bounded integral $K = \int_{A \times [-\infty, \infty]} f_0 > 0$.

Proposition 17 *Let $B = [-b, b]$ and $y_1^u = -u/2$, $y_2^u = u/2$. Consider the conditional density f_b induced⁶ by f_0 on $A \times B$. Assume parties use the mean voter rule. There is $\bar{b} > 0$ such that if $b > \bar{b}$ and $\bar{b} \leq u \leq b$ then, the model $(A, B, f_b, y_1^u, y_2^u, P_1^u, P_2^u)$ has a unique equilibrium.*

It follows that if parties are *ex ante* identical, then this unique equilibrium is convergent. The proof is provided in the ‘‘Appendix’’. The Assumption in Proposition 17 can be interpreted as saying that if the population is not too concentrated around the line $y = 0$, then the equilibrium is unique. On the other hand, Proposition 13 shows that, if b is small enough, then there are divergent equilibria, even if parties are *ex*

⁶ Note that we parametrize f_b by the endpoints of $B = [-b, b]$.

ante identical. The following example illustrates exactly this situation. As $|y_2 - y_1|$ increases, the proposals of the parties converge to a unique equilibrium, which then becomes stable.

Example 18 Let $A = [0, 1]$, $B = [-b, b]$. We assume that agents are uniformly distributed on $A \times B$ and parties aggregate the preferences of their members using the mean voter rule. As before, we let

$$y_1 = -\frac{u}{2}, \quad y_2 = \frac{u}{2}$$

We claim that for $u > 1/2b$, there is a unique convergent equilibrium. Let $x = (x_1, x_2)$. Recall that

$$z(t; x) = \frac{2t(x_2 - x_1) + x_1^2 - x_2^2 + y_1^2 - y_2^2}{2(y_1 - y_2)} = \frac{(x_1 - x_2)(2t - x_1 - x_2)}{2u}$$

After some straightforward computations, we get that

$$g_i(t; x) = \frac{2tx_i - 2tx_j + u - x_i^2 + x_j^2}{2u}, \quad i, j = 1, 2, \quad i \neq j$$

In the following computations, we assume that u is large enough so that the graph of the line $z(t; x)$ does not intersect the lines $y = \pm b$, for $t, x_1, x_2 \in [0, 1]$. Since

$$z(0; x) = \frac{x_2^2 - x_1^2}{2u}, \quad z(1; x) = \frac{(x_2 - x_1)(x_1 + x_2 - 2)}{2u}$$

this can be guaranteed as long as $u > 1/2b$. Under this assumption, we obtain that⁷

$$\begin{aligned} \phi_1(x) &= \frac{1}{6} \left(3 - \frac{x_2 - x_1}{2bu - (x_1 - x_2)(x_1 + x_2 - 1)} \right), \\ \phi_2(x) &= \frac{1}{6} \left(3 + \frac{x_2 - x_1}{2bu + (x_1 - x_2)(x_1 + x_2 - 1)} \right) \end{aligned}$$

The equilibria, being the fixed points of ϕ , are determined by the equations $x_i = \phi_i(x)$, $i = 1, 2$. A straightforward computation shows that these equations are equivalent to the following system of equations

$$\begin{aligned} 0 &= 6bu(1 - 2x_1) - (x_2 - x_1)(4 - 3x_2 + 3x_1(2x_1 + 2x_2 - 3)) \\ 0 &= 6bu(1 - 2x_2) + (x_2 - x_1)(4 - 9x_2 + 6x_2^2 + 3x_1(2x_2 - 1)) \end{aligned}$$

⁷ The expressions of ϕ_1, ϕ_2 are much more complicated for values of $t, x_1, x_2 \in [0, 1]$ for which the graph of $z(t; x)$ intersects the lines $y = \pm b$.

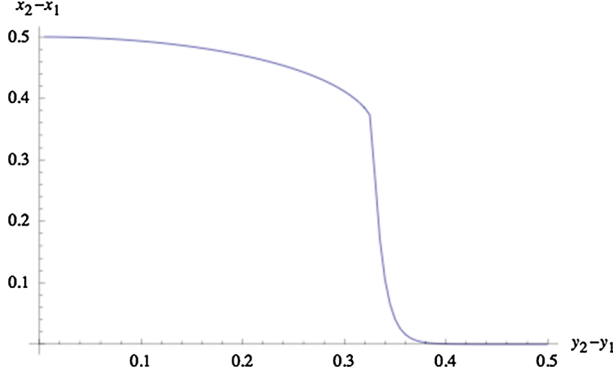


Fig. 2 The divergent equilibrium with $x_1 < x_2$. Mean voter rule

Adding and subtracting the above two equations, we obtain the following system of equations

$$\begin{aligned} 0 &= (x_1 + x_2 - 1) \left((x_1 - x_2)^2 - 2bu \right) \\ 0 &= (x_1 - x_2)(4 - 6bu + 3x_1^2 - 6x_1(1 - x_2) - 3x_2(2 - x_2)) \end{aligned}$$

We see that $x_1 = x_2 = 1/2$ is a solution. If $x_1 \neq x_2$, then we must have that

$$u = \frac{(x_2 - x_1)^2}{2b}$$

So, for $0 \leq x_1, x_2 \leq 1$; the function $\frac{(x_2 - x_1)^2}{2b}$ attains the maximum value $1/2b$ at the points $x_1 = 0, x_2 = 1$ and $x_1 = 1, x_2 = 0$. Thus, for $u > 1/2b$, the unique solution is $x_1 = x_2 = 1/2$.

We can illustrate this numerically. Let $A = B = [0, 1]$ and let the population distribution be uniform on $[0, 1] \times [0, 1]$, so that $f(\alpha, \beta) = 1$ for $\alpha, \beta \in [0, 1]$ and $f(\alpha, \beta) = 0$ otherwise. We focus on the equilibrium⁸ with $x_1 < x_2$. We fix $y_1 = 0.2$ and take values of y_2 running from 0.21 to 0.8 in steps of 0.01.

Figure 2 shows how the divergent equilibrium is such that the difference between the two policies, $|x_2^* - x_1^*|$, is decreasing as $|y_2 - y_1|$ increases. The ordinate shows the different values of y_1 and the abscissas the corresponding values of $|x_2^* - x_1^*|$. We see that there is a negative relation between the value of y_2 (which increases the value of the difference $|y_2 - y_1|$) and the difference $|x_2^* - x_1^*|$.

4 The median voter rule

It is possible to use the same techniques that we used for the median voter rule to establish similar bounds for other rules. Suppose $n = 1$ (so that $X = [x_0, x_1]$ is a

⁸ Of course, there is a corresponding symmetric equilibrium with $x_1 > x_2$.

compact interval) and parties use the median⁹ voter rule. That is, for each $\mu \in \Sigma$ the mapping P^m assigns $P^m(\mu)$ defined implicitly by the following equation.

$$\mu(\{\alpha \in A : \alpha \leq P^m(\mu)\}) = \frac{\mu(A)}{2}$$

We assume now that parties use the aggregation rule $P_1^m = P_2^m = P^m$. As before, we define $\phi : X \times X \rightarrow X \times X$ by $\phi = (\phi_1, \phi_2)$, with $\phi_i = P^m \circ \sigma_i$, $i = 1, 2$. Clearly, the equilibria of this model are the fixed points of the mapping ϕ , given implicitly by the equations

$$\int_{x_0}^{\phi_j(x)} g_j(\alpha; x) d\alpha = \frac{1}{2} \int_{x_0}^{x_1} g_j(\alpha; x) d\alpha, \quad j = 1, 2 \quad (9)$$

The following result is proved in the “Appendix”.

Lemma 19 *The map $\phi : X \times X \rightarrow X \times X$ is continuous and differentiable.*

For a given a distribution $f : A \times B \rightarrow \mathbb{R}$ of the population we denote by $m_j(x) = P_j^m(\sigma_j(x))$ the median of the distribution $\sigma_j(x)$ for $j = 1, 2$. Recall that, when parties make the same proposal $x = (x_1, x_2)$ with $x_1 = x_2$, we have that $\sigma_1(x)$ and $\sigma_2(x)$ (and hence $m_1 = m_1(x)$ and $m_2 = m_2(x)$) are independent of x . There is a convergent equilibrium iff $m_1(x) = m_2(x)$ for $x = (x_1, x_2)$ such that $x_1 = x_2$. However, as remarked for the average rule, there is no reason why we should expect that $m_1 = m_2$, even if it happens that parties aggregate the preferences of their members using the same decision rule $P_1 = P_2$. As with the average aggregation rule, we are interested in showing that whenever y_1 and y_2 are close enough and it happens that $m_1 = m_2 = m$ (for whatever reasons), the convergent equilibrium (m, m) is unstable. So, let us assume that (m, m) is a convergent equilibrium. That is

$$\int_{\{\alpha \in A : \alpha \leq m\}} g_j(\alpha) d\alpha = \frac{\lambda_j}{2} \quad j = 1, 2$$

with λ_1, λ_2 defined in (5). As before, we will establish conditions under which inequality (7) holds when we use the maps ϕ_1, ϕ_2 defined in (9). Let

$$f_1(m) = \int_{x_0}^{\frac{y_1+y_2}{2}} f(m, \beta) d\beta$$

The following result is proved in the “Appendix”.

⁹ [Ansolabehere et al. \(2012\)](#) assume that parties represent the median of their elected officials.

Proposition 20 *Let $n = 1$. Suppose that parties use the median voter rule and that $m_1 = m_2 = m$. If*

$$\frac{1}{f_1(m)} \left(\int_{x_0}^{x_1} (\alpha - m) f \left(\alpha, \frac{y_1 + y_2}{2} \right) d\alpha - 2 \int_{x_0}^m (\alpha - m) f \left(\alpha, \frac{y_1 + y_2}{2} \right) d\alpha \right) > |y_2 - y_1| > 0 \quad (10)$$

then there exists a divergent equilibrium.

As a consequence, we have the following.

Corollary 21 *Let $n = 1$. Suppose the parties are ex ante identical, use the median voter rule and the bound (20) holds. Then, there exists a divergent equilibrium.*

It should be noted that when the density of citizens' ideological viewpoints at the median point of the whole distribution is $f_1(m) = 0$, then the bound on $|y_2 - y_1|$ explodes, as minor changes of policies cause the intra-party medians to move at an infinite rate.¹⁰

Example 22 Take again $n = 1$ and the population distribution be uniform on $[0, 1] \times [0, 1]$ so that $f(\alpha, \beta) = 1$ and $\lambda_1 = \lambda_2 = 1/2$. Suppose that both parties use the median voter rule P^m . In this case, the bound for $|y_2 - y_1|$ implied by Proposition 20 is easily computed to be equal to $1/2$. That is, as long as $0 < |y_2 - y_1| < 1/2$ there exists a divergent equilibrium.

We see that, with the uniform distribution of citizens over $A \times B = [0, 1] \times [0, 1]$, the bound for $|y_2 - y_1|$ implied by the median voter rule is weaker than that for the mean voter rule (for which it is $\frac{1}{3}$). Therefore, as the policy difference between parties induces ideologically skewed memberships within each party, the medians move toward the edges faster than the means.

As an example satisfying this inequality take $y_1 = 0.55$ and $y_2 = 0.20$. It is not difficult to show that the unique divergent equilibrium has $x_1 = 0.389972$, $x_2 = 0.717886$.

It is straightforward to prove the analogue of Proposition 16 for the median voter rule. The statement of the result is exactly the same except, of course, that one assumes now that parties use the median voter rule. We leave the details to the interested reader.

Likewise, one might obtain similar results to those in Propositions 13 and 20 that guarantee the existence of divergent equilibria when parties use distinct rules. Of course, the result is trivially true if there is no convergent equilibrium. For example, if party 1 uses the mean voter rule, while party 2 uses the median voter rule a convergent equilibrium only exists if the mean equals the median for the overall population distribution $f_1(\alpha)$, i.e., if $m = \mu$. However, even for the case when $m = \mu$ we can guarantee

¹⁰ Though, strictly speaking, the function ϕ is not differentiable (not even necessarily continuous) in this case, the instability of the convergent equilibrium and the consequent existence of a divergent equilibrium can be easily shown using standard approximation techniques.

existence of a divergent equilibria for $|y_2 - y_1|$ small enough. In fact, using inequality 7, we can establish that such equilibria exist in this case whenever the following condition holds.

$$0 < |y_2 - y_1| < \frac{1}{\lambda_1} \int_A (\alpha - \mu)^2 f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha \\ + \frac{1}{1 - \lambda_1} \frac{1}{f_1(\mu)} \int_{\{\alpha: \alpha \leq \mu\}} (\mu - \alpha) f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha$$

4.1 Policy divergence and non-ideological characteristics for the median voter rule

In this section, we consider again the convergence/divergence of the equilibria as $y_2 - y_1$ varies. We will show that the analogue of Example 18 holds also when parties aggregate the preferences of their members via the median voter rule.

Example 23 Let $A = [0, 1]$, $B = [-b, b]$ and assume that agents are uniformly distributed on $A \times B$ and parties aggregate the preferences of their members using the median voter rule. As in Example 18, we let

$$y_1 = -\frac{u}{2}, \quad y_2 = \frac{u}{2}$$

Recall that

$$z(t; x) = \frac{(x_1 - x_2)(2t - x_1 - x_2)}{2u}$$

and

$$g_1(t; x) = \frac{2tx_1 - 2tx_2 + u - x_1^2 + x_2^2}{2u} \\ g_2(t; x) = \frac{2tx_2 - 2tx_1 + u + x_1^2 - x_2^2}{2u}$$

We assume again that $u \geq 1/2b$ so that the graph of $z(t; x)$ does not intersect the lines $y = \pm b$. The expressions of ϕ_1, ϕ_2 are fairly cumbersome and we skip them here. We remark though that one can check that the solutions to $\phi_i(x_1, x_2) = x_i$, $i = 1, 2$ are $(x_1^*, x_2^*) = (1 - \sqrt{1 - bu}, 1 + \sqrt{1 - bu})$, $(x_1^*, x_2^*) = (1 + \sqrt{1 - bu}, 1 - \sqrt{1 - bu})$ and $(x_1^*, x_2^*) = (\frac{1}{2}, \frac{1}{2})$. Of course, the first two equilibria are only valid if $bu \leq 1$, $1 - \sqrt{1 - bu} \geq -b$ and $1 + \sqrt{1 - bu} \leq b$. Thus, for b fixed the only equilibrium is the convergent one, as long as $bu > 1$.

Figure 3 graphs numerically the fixed points corresponding to the idiosyncratic issues. In there, we take $A = B = [0, 1]$ and the population is distributed uniformly on $[0, 1] \times [0, 1]$. We fix $y_1 = 0.2$ and take values of y_2 running from 0.21 to 0.8 in steps of 0.01, so $y_2 - y_1$ varies from 0.01 to 0.5. The graph plots the values of the equilibria (abscissas) versus the values of $y_2 - y_1$ (ordinates). The blue and red lines represent the values of the equilibria, x_1 (red) and x_2 (blue).

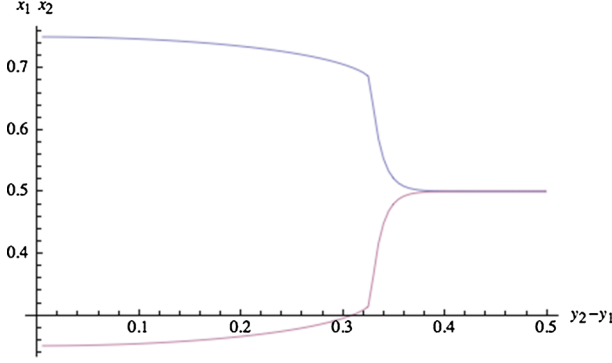


Fig. 3 The unique divergent equilibrium with $x_1 < x_2$. Median Voter Rule

The intuition is the same as for the mean rule. If the non-ideological characteristics y_1 and y_2 are very different, the memberships of parties are basically determined by such characteristics.

In this case, and given that the distributions of α and β are independent, both parties will have similar mean and median values of α . Examples 15 and 22 provide an extreme case of this type of situation where the equilibrium policy characteristics of both parties coincide.

5 Conclusions and further research

In this paper, we have introduced a model in which the citizens' party choice is determined by both the ideological difference between the parties and the unobserved non-ideological attitudes. As the membership choice is only incompletely determined by the observed policy proposals, it may be interpreted as stochastic from the point of view of the parties. The party membership, in turn, determines party policy stances by means of intra-party preference aggregation rules.

We have especially focused on the important cases in which parties aggregate preferences by choosing ideal points of their mean or median voters. Parties are perceived by citizens as 'similar' in the sense that the non-policy difference is small compared to the mean of the agents' preferences in the ideological space. In this context and with two parties we show that we are guaranteed existence of divergent equilibria, even in an *ex ante* symmetric model. In this sense, the present stochastic model shows continuity with the deterministic endogenous platform model studied earlier in [Gomberg et al. \(2004\)](#).

It should be noted that a similar stochastic preference model has been previously considered in [Caplin and Nalebuff \(1997\)](#). However, the authors of that paper believed that approach was, in a certain sense, simply imposing policy divergence exogenously, rather than having it emerge endogenously in the model. In fact, they were worried that as the stochastic preference component would become smaller, equilibrium policy divergence would itself disappear. This observation, in fact, motivated their 'index theory' approach, which we have also utilized since. What we show in this paper,

however, is that, under the assumptions of the model, properly divergent equilibria are guaranteed to exist even as the stochastic preference component becomes weaker.

The model provides what we believe are original insights pertaining to the level of party polarization¹¹. Under certain conditions on the distribution of preferences, if parties are very different in their non-policy characteristics, their policy proposals will be very similar.

It remains to consider how the results extend to increasing the number of parties, as well as considering different party decision-making rules.

6 Appendix: Proofs

Proof of Lemma 5 We will show that the mappings $\sigma_1, \sigma_2 : X \times X \rightarrow \Sigma$ are Fréchet differentiable (which, of course, implies continuity). We do the proof for $i = 1$. Let $x = (x_1, x_2) \in X \times X$. Note that

$$\frac{\partial g_1}{\partial x_1} = f(\alpha, z(\alpha; x)) \frac{\alpha - x_1}{y_2 - y_1}$$

and

$$\frac{\partial g_1}{\partial x_2} = f(\alpha, z(\alpha; x)) \frac{x_2 - \alpha}{y_2 - y_1}$$

From the above expressions, the Fréchet derivative can be easily computed. We use the notation $e = (h, k)$ and $\|e\| = \sqrt{h^2 + k^2}$. Fix $x \in X \times X$. A simple computation shows that

$$\begin{aligned} z(\alpha; x + e) - z(\alpha; x) &= \frac{h(\alpha - x_1) + k(x_2 - \alpha) - (k^2 - h^2)/2}{y_2 - y_1} \\ &= \frac{(h - k)\alpha - x_1 h + x_2 k}{y_2 - y_1} + \frac{h^2 - k^2}{2(y_2 - y_1)} \end{aligned}$$

Note that there is $M > 0$ such that

$$\frac{|z(\alpha; x + e) - z(\alpha; x)|}{\|e\|} \leq M$$

for any $\alpha \in A$ and e such that $\|e\| \leq 1$. Let $\varepsilon > 0$. Note that

$$g_1(\alpha; x + e) - g_1(\alpha; x) = \int_{z(\alpha; x)}^{z(\alpha; x + e)} f(\alpha, \beta) d\beta$$

We are ready now to compute $D\sigma(e; x) : \mathbb{R}^2 \rightarrow C(A)$, the derivative of σ at the point x . We continue to use the notation $e = (h, k)$. We will show that $D\sigma(e; x)(\alpha)$ takes the following value

¹¹ See [Fauli-Oller et al. \(2003\)](#) for a different view on party polarization.

$$\begin{aligned}
D\sigma(e; x)(\alpha) &= \frac{f(\alpha, z(\alpha; x))}{y_2 - y_1} (h(\alpha - x_1) + k(x_2 - \alpha)) \\
&= \int_{z(\alpha; x)}^{z(\alpha; x+e)} f(\alpha, z(\alpha; x)) d\beta - \frac{k^2 - h^2}{2|y_2 - y_1|}
\end{aligned}$$

Note that $D\sigma(e; x)(\alpha)$, as defined in the above equation, is linear in e and continuous in all the variables. Note that,

$$\begin{aligned}
&g_1(\alpha; x + e) - g_1(\alpha; x) - D\sigma(e; x)(\alpha) \\
&= \int_{z(\alpha; x)}^{z(\alpha; x+e)} (f(\alpha, \beta) - f(\alpha, z(\alpha; x))) d\beta + \frac{k^2 - h^2}{2|y_2 - y_1|}
\end{aligned}$$

Since z is continuous and $f(\alpha, \beta)$ is uniformly continuous in $A \times [z(\alpha; x), z(\alpha; x+e)]$, given $\varepsilon > 0$, there is a $0 < \delta < 1$ such that if $\|e\| \leq \delta$, then

$$|f(\alpha, \beta) - f(\alpha, z(\alpha; x))| \leq \frac{\varepsilon}{M}$$

Thus, as long as $\|e\| \leq \delta$ we have that

$$|g_1(\alpha; x + e) - g_1(\alpha; (x)) - D\sigma(e; x)(\alpha)| \leq \frac{\varepsilon}{M} |z(\alpha; x + e) - z(\alpha; x)| + \frac{k^2 + h^2}{2|y_2 - y_1|}$$

So, for any $\alpha \in A$, the following holds

$$\frac{|g_1(\alpha; x + e) - g_1(\alpha; (x)) - D\sigma(e; x)(\alpha)|}{\|e\|} \leq \varepsilon + \frac{k^2 + h^2}{2\|e\|} = \varepsilon + \frac{\|e\|}{2|y_2 - y_1|}$$

Hence, we have that

$$\sup_{\alpha \in A} \frac{|g_1(\alpha; x + e) - g_1(\alpha; (x)) - D\sigma(e; x)(\alpha)|}{\|e\|} \leq \varepsilon + \frac{\|e\|}{2|y_2 - y_1|}$$

Therefore,

$$\frac{\|g_1(\alpha; x + e) - g_1(\cdot; x) - D\sigma(e; x)\|}{\|e\|} \leq \varepsilon + \frac{\|e\|}{2|y_2 - y_1|}$$

Since $\varepsilon > 0$ is arbitrary, we see that

$$\lim_{\|e\| \rightarrow 0} \frac{\|\sigma_1(x + e) - \sigma_1(x) - D\sigma(e; x)\|}{\|e\|} = 0$$

and the Lemma is proved.

Proof of Lemma 12 Lemma 12 is consequence of Lemma 5 and of Lemma 24 below.

Lemma 24 The map $P^\mu : \Sigma \times \Sigma \rightarrow A \times A$ is continuous and differentiable.

Proof of Lemma 24 For each $i = 1, 2$, the maps $v_i \mapsto v_i(A)$ and $v_i \mapsto \int_A \alpha d v_i(\alpha)$ are linear. It is easy to see that they are also continuous. For example, given $\varepsilon > 0$, let

$$\delta = \frac{\varepsilon}{\int_A \alpha d\alpha}$$

If $\sup\{|f(\alpha) - g(\alpha)| : \alpha \in A\} \leq \delta$ then

$$\left| \int_A \alpha f(\alpha) d\alpha - \int_A \alpha g(\alpha) d\alpha \right| \leq \int_A \alpha |f(\alpha) - g(\alpha)| d\alpha \leq \delta \int_A \alpha d\alpha = \varepsilon$$

So $v_i \mapsto \int_A \alpha d v_i(\alpha)$ is continuous. The proof that $v_i \mapsto v_i(A)$ is continuous is similar. Hence, it follows that the maps $v_i \mapsto v_i(A)$ and $v_i \mapsto \int_A \alpha d v_i(\alpha)$ are differentiable.

Since for every $i = 1, 2$ we have that $v_i(A) \neq 0$, the mapping $v_i \mapsto \frac{\int_A \alpha d v_i(\alpha)}{v_i(A)}$ is also differentiable. Therefore, so is P^μ .

For the proof of Proposition 13, we will make use of the following result.

Lemma 25

$$\lim_{x \rightarrow (\mu, \mu)} \frac{\partial \phi_j}{\partial x_i}(x) = \frac{1}{\lambda_j} \int_A (\alpha - \mu) \frac{\partial g_j(\alpha; (\mu, \mu))}{\partial x_i} d\alpha$$

Proof of Lemma 25 Note that

$$\begin{aligned} \frac{\partial \phi_j}{\partial x_i}(x) &= \frac{1}{(\int_A g_j(\alpha; x) d\alpha)^2} \left(\int_A \alpha \frac{\partial g_j}{\partial x_i}(\alpha; (\mu, \mu)) d\alpha \int_A g_j(\alpha; x) d\alpha \right. \\ &\quad \left. - \int_A \alpha g_j(\alpha; x) d\alpha \int_A \frac{\partial g_j}{\partial x_i}(\alpha; (\mu, \mu)) d\alpha \right) \\ &= \frac{1}{(\int_A g_j(\alpha; x) d\alpha)} \int_A (\alpha - \phi_j(x)) \frac{\partial g_j}{\partial x_i}(\alpha; (\mu, \mu)) d\alpha \end{aligned}$$

Thus,

$$\lim_{x \rightarrow (\mu, \mu)} \frac{\partial \phi_j}{\partial x_i}(x) = \frac{1}{\lambda_j} \int_A (\alpha - \mu) \frac{\partial g_j}{\partial x_i}(\alpha; (\mu, \mu)) d\alpha$$

and the Lemma follows.

Proof of Proposition 13 Recall that, given the proposals $x = (x_1, x_2)$ of the parties, we use the notation

$$g_j(\alpha; x) = \int_{\{(\alpha, \beta) : \|(x_j, y_j) - (\alpha, \beta)\| \geq \|(x_i, y_i) - (\alpha, \beta)\|, i \neq j\}} f(\alpha, \beta) d\beta$$

when we want to make explicit the dependence of the density functions that describe the induced population partitions on the policies proposed by the parties. We have seen in the proof of Lemma 5 that

$$\frac{\partial g_1(\alpha; x)}{\partial x_1} = f(\alpha, z(\alpha; x)) \partial_1 z(t; x)|_{t=\alpha} = f(\alpha, z(\alpha; x)) \frac{\alpha - x_1}{y_2 - y_1}$$

Furthermore,

$$\frac{\partial g_1(\alpha; x)}{\partial x_2} = f(\alpha, z(\alpha; x)) \partial_2 z(t; x)|_{t=\alpha} = f(\alpha, z(\alpha; x)) \frac{x_2 - \alpha}{y_2 - y_1} \quad (11)$$

which implies that

$$\frac{\partial g_1}{\partial x_1}(\alpha; (\mu, \mu)) = -\frac{\partial g_1}{\partial x_2}(\alpha; (\mu, \mu)) = f\left(\alpha, \frac{y_1 + y_2}{2}\right) \frac{\alpha - \mu}{y_2 - y_1}$$

Since for party 2 the relevant population density is

$$g_2(\alpha; x) = \int_{z(\alpha; x)}^{\infty} f(\alpha, \beta) d\beta$$

we get that

$$\begin{aligned} \frac{\partial g_2}{\partial x_2}(\alpha; (\mu, \mu)) &= -\frac{\partial g_2}{\partial x_1}(\alpha; (\mu, \mu)) = \frac{\partial g_1}{\partial x_1}(\alpha; (\mu, \mu)) \\ &= f\left(\alpha, \frac{y_1 + y_2}{2}\right) \frac{\alpha - \mu}{y_2 - y_1} \end{aligned} \quad (12)$$

To ease the notation, we will write $g_i = g_i(\alpha; (\mu, \mu))$. Applying Lemma 25 and using the above formulae for $\frac{\partial g_i}{\partial x_i}$, we have to establish conditions under which

$$\begin{aligned} \lim_{x \rightarrow (\mu, \mu)} |B(x)| &= \left| \frac{1}{\lambda_1} \int_A (\alpha - \mu) \frac{\partial g_1}{\partial x_1} d\alpha - 1 \frac{1}{\lambda_1} \int_A (\alpha - \mu) \frac{\partial g_1}{\partial x_2} d\alpha - 1 \right| \\ &\quad \left| \frac{1}{\lambda_2} \int_A (\alpha - \mu) \frac{\partial g_2}{\partial x_1} d\alpha - 1 \frac{1}{\lambda_2} \int_A (\alpha - \mu) \frac{\partial g_2}{\partial x_2} d\alpha - 1 \right| \\ &= \left| \frac{1}{\lambda_1} \int_A (\alpha - \mu) \frac{\partial g_1}{\partial x_1} d\alpha - 1 \frac{1}{\lambda_1} \int_A (\alpha - \mu) \frac{\partial g_1}{\partial x_1} d\alpha - 1 \right| \\ &\quad \left| \frac{1}{\lambda_2} \int_A (\alpha - \mu) \frac{\partial g_1}{\partial x_1} d\alpha - 1 \frac{1}{\lambda_2} \int_A (\alpha - \mu) \frac{\partial g_1}{\partial x_1} d\alpha - 1 \right| \\ &= 1 - \frac{1}{\lambda_1 \lambda_2} \int_A (\alpha - \mu) \frac{\partial g_1}{\partial x_1} d\alpha \\ &= 1 - \frac{1}{\lambda_1 \lambda_2 (y_2 - y_1)} \int_A (\alpha - \mu)^2 f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha < 0 \end{aligned}$$

which clearly holds if

$$|y_2 - y_1| < \frac{1}{\lambda_1 \lambda_2} \int_A (\alpha - \mu)^2 f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha$$

The Proposition follows from the above inequality.

Proof of Proposition 16 Fix a pair of $x_1 \neq x_2$ and for any $s \in [0, 1]$ define $y_1^s = y_1 + s(y_2 - y_1)$. Note that as $s \rightarrow 0$ the induced partition hyperplane will converge to perfect sorting in the policy dimension, as in Remark 1. This, of course, implies that the induced population partitions for $g_1^s(\alpha; x)$, $g_2^s(\alpha; x)$ converge uniformly to those given in Remark 1. Hence, for any $x \notin \Delta$ the induced measures $\sigma_1^s(x)$, $\sigma_2^s(x)$ will converge uniformly in L^1 as $s \rightarrow 0$ to the measures obtained in Remark 1. (Note that this convergence is not uniform for $x \in X \times X$, as, in fact, the functions will not converge on the diagonal.) It follows that the stable fixed points of ϕ^s converge to some stable fixed points of the limiting model.

Now, in the proof of Proposition 13 we have seen that

$$\begin{aligned} |B^s(\mu, \mu)| &= 1 - \frac{1}{\lambda_1 \lambda_2 (y_2^s - y_1^s)} \int_A (\alpha - \mu)^2 f\left(\alpha, \frac{y_1^s + y_2^s}{2}\right) d\alpha \\ &= 1 - \frac{1}{s} \left(\frac{1}{2\lambda_1 \lambda_2 (y_2 - y_1)} \int_A (\alpha - \mu)^2 f(\alpha, y_1 + y_2) d\alpha \right) \end{aligned}$$

The term inside the brackets does not depend on s . Hence, there is a $\varepsilon > 0$ such that $|B^s(\mu, \mu)| < 0$ for any $x \in X \times X$ and $0 < s < \varepsilon$. It follows that the stable fixed points of ϕ^s cannot converge to the diagonal.

Proof of Proposition 17 Since $y_1'' = -\frac{u}{2}$, $y_2'' = \frac{u}{2}$, we have that $\frac{y_1'' + y_2''}{2} = 0$. To make explicit the dependence on u , we will write $z(t; x; u) = z(t; x) = \frac{(x_1 - x_2)(2t - x_1 - x_2)}{2u}$, $\phi_j(x; u) = \phi_j(x)$ and let $g_i(\alpha; x; u) = g_i(\alpha; x)$, $i = 1, 2$, defined by (3) with f replaced by f_b . Note that

$$|z(t; x; u)| \leq \frac{1}{u}$$

for every $x = (x_1, x_2) \in A \times A$, $t \in A$. Hence, $\lim_{u \rightarrow +\infty} z(t; x; u) = 0$ and

$$\lim_{u \rightarrow \infty} \int_A g_i(\alpha; x; u) d\alpha = \lambda_i > 0, \quad i = 1, 2$$

uniformly on $x \in A \times A$. That is, there is a real number $b_1 > 0$ such that if $u \geq b_1$, then

$$\int_A g_i(\alpha; x; u) d\alpha \geq \frac{\lambda_i}{2}$$

for every $x \in X \times X$. Let M be the supremum of f_0 on the set $A \times [-\infty, \infty]$. From the proof of Lemma 24, we see that

$$\left| \frac{\partial g_j(\alpha; x; u)}{\partial x_i} \right| = f(\alpha, z(\alpha; x)) \frac{|\alpha - x_i|}{|y_2 - y_1|} = f(\alpha, z(\alpha; x)) \frac{|\alpha - x_i|}{u} \leq M \frac{|\alpha - x_i|}{u}$$

And, since $\alpha, \phi_j(x; u)$ and x_i belong to $A = [0, 1]$ we have that $|\alpha - \phi_j(x; u)| \leq 1$ and $|\alpha - x_i| \leq 1$. Hence,

$$\left| \frac{\partial \phi_j(x; u)}{\partial x_i} \right| \leq \frac{1}{(\int_A g_j(\alpha; x) d\alpha)} \int_A |\alpha - \phi_j(x)| \left| \frac{\partial g_j(\alpha; x; u)}{\partial x_i} \right| d\alpha \leq \frac{M}{u \lambda_j}$$

for every $x \in X \times X$ and $u > b_1$. It follows that there is $\bar{b} \geq b_1$ such that $\left| \frac{\partial \phi_j(x; u)}{\partial x_i} \right| < 1/4$ for every $u \geq \bar{b}$ and for every $x \in X \times X$.

Let $b \geq u \geq \bar{b}$. The proof of Lemma 24 shows that $\det(B(x)) = 1 + \partial_1 \phi_1(x; u) \partial_2 \phi_2(x; u) - \partial_1 \phi_2(x; u) \partial_2 \phi_1(x; u) - \partial_1 \phi_2(x; u) - \partial_2 \phi_2(x; u) > 0$. Since the sum of the indices of the fixed points must add to the Euler Characteristics of the rectangle $A \times B$, which is $+1$, there can be at most a fixed point.

Proof of Lemma 19 Let $j = 1, 2$ and consider the function $F_j : X \times X \times X \rightarrow \mathbb{R}$ defined by

$$F(x_1, x_2, y) = \int_{-\infty}^y g_j(\alpha; x) d\alpha$$

Since $f(\alpha, \beta)$ is equivalent to Lebesgue measure, for each $x = (x_1, x_2) \in X \times X$, the equation

$$F(x_1, x_2, y) = \frac{\lambda_j}{2}$$

has a unique solution $y = \phi_j(x)$ in the interior of X . And since

$$\frac{\partial F}{\partial y} = g_j(y; x) > 0$$

we may apply the implicit function theorem to conclude that there is an open neighborhood U of x in $X \times X$, an open neighborhood V of y in X and a continuously differentiable function $\phi_j : U \rightarrow V$ such that

$$F(x_1, x_2, \phi_j(x)) = \frac{\lambda_j}{2}$$

Hence, the function $\phi = (\phi_1, \phi_2) : V \times V \rightarrow X$ is continuously differentiable.

Proof of Proposition 20 Differentiating ϕ implicitly with respect to x_i in Equation (9), we obtain

$$g_j(\phi_j(x); x) \frac{\partial \phi_j}{\partial x_i}(x) + \int_{-\infty}^{\phi_j(x_j)} \frac{\partial g_j(\alpha; x)}{\partial x_i} d\alpha = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial g_j(\alpha; x)}{\partial x_i} d\alpha$$

Taking the limit of the expression as $x \rightarrow (m, m)$, we obtain

$$f_1(m) \frac{\partial \phi_j}{\partial x_i}(m) + \int_{-\infty}^m \frac{\partial g_j}{\partial x_i}(\alpha; m) d\alpha = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial g_j}{\partial x_i}(\alpha; m) d\alpha \quad (13)$$

with

$$f_1(m) = \int_{-\infty}^{\frac{y_1+y_2}{2}} f(m, \beta) d\beta$$

Now, taking into account Eqs. (11) and (12) the formulas for $\frac{\partial g_j(\alpha; x)}{\partial x_i}$, we see that

$$\frac{\partial \phi_1}{\partial x_1}(m) = \frac{\partial \phi_2}{\partial x_2}(m) = -\frac{\partial \phi_2}{\partial x_1}(m) = -\frac{\partial \phi_1}{\partial x_2}(m)$$

Thus, the determinant in (7) becomes,

$$B(x) = \begin{vmatrix} \frac{\partial \phi_1}{\partial x_1}(m) - 1 & \frac{\partial \phi_1}{\partial x_2}(m) \\ \frac{\partial \phi_2}{\partial x_1}(m) & \frac{\partial \phi_2}{\partial x_2}(m) - 1 \end{vmatrix} = \begin{vmatrix} \frac{\partial \phi_1}{\partial x_1}(m) - 1 & -\frac{\partial \phi_1}{\partial x_1}(m) \\ -\frac{\partial \phi_1}{\partial x_1}(m) & \frac{\partial \phi_1}{\partial x_1}(m) - 1 \end{vmatrix} = 1 - 2 \frac{\partial \phi_1}{\partial x_1}(m)$$

Finally, from equations (13), (11) and (12) we get that

$$\begin{aligned} \frac{\partial \phi_1}{\partial x_1}(x) = & \frac{1}{(y_2 - y_1) f_1(m)} \left(\frac{1}{2} \int_{-\infty}^{\infty} (\alpha - m) f \left(\alpha, \frac{y_1 + y_2}{2} \right) d\alpha \right. \\ & \left. - \int_{-\infty}^m (\alpha - m) f \left(\alpha, \frac{y_1 + y_2}{2} \right) d\alpha \right). \end{aligned}$$

And the proposition follows.

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